Conveying ignorance
Ignorance inferences with superlative numeral modifiers

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This paper investigates the inferences of ignorance that come with superlative numerals, like *at least n* and *at most n*. It argues that these are better understood as primary implicatures that are derived in a neo–Gricean framework together with a standard epistemic logic. In doing so, the paper supports the view of superlative numerals first advanced by Büring (2007) and further elaborated by Schwarz (2013), where the ignorance inferences that they convey are equal to those of a disjunctive expression, of the form *exactly n or more/less than n*.

1. Introduction

Superlative Modifiers (SMs henceforth) like *at most* and *at least* often convey that the speaker is uncertain about some exact value (Geurts and Nouwen 2007; Nouwen 2010):

(1) a. #I have at most two daughters.
 b. #I have at least five fingers.

The examples in (1) are odd. The epistemic competence commonly assumed when we talk about progeny or our own body is at odds with the presence of SMs and their incompatibility with full knowledge. The resulting effect has been dubbed the 'epistemic effect' of SMs: It is triggered by the fact that SMs convey an inference of ignorance (II henceforth) on the speaker as to what the exact amount is. The contrast with (2), where no speaker knowledge is assumed, is clear:

(2) a. Bill has at most two daughters.
 b. That caterpillar has at least twenty legs.

That SMs trigger IIs is uncontroversial. The controversy is about how exactly IIs arise and whether this is the fruit of the semantic properties of SMs (Geurts and Nouwen 2007, Nouwen 2010) or whether it is the result of a pragmatic process (Büring 2007, Schwarz 2013, 2014, Coppock and Brochhagen 2013b, Mayr 2013, Nouwen 2015, Kennedy 2015). In addition, more

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surprisingly perhaps, the literature is lacking in a debate about what the exact form of these IIs is. Given a sentence containing an SM, how much can be inferred about what the speaker is ignorant about? Despite the efforts devoted to craft a theory of SMs that accounts for all their relevant semantic and pragmatic properties, this question has not yet been thoroughly scrutinized. As a consequence, there is some unclarity with respect to what exactly these IIs have to be. Clarifying this question is important not just because we want a descriptively adequate theory of SMs, but also because carefully defining the form of these IIs provides further criteria for evaluating theories of SMs.

Different formulations of IIs may make different predictions with respect to what can be inferred about what the speaker is ignorant about. For instance, consider the following sentences:

(3) a. At least three boys left. [Mayr 2013:158]

b. Bill is at least an assistant professor. [Coppock and Brochhagen 2013b:10]

About (3a), Mayr says that it ‘seems to have the ignorance inference that for any number *n* larger than three, the speaker fails to believe that n–many boys left’. Similarly, Coppock and Brochhagen suggest that the meaning of (3b) ‘can be expressed as a disjunction over the answers that are at least as strong: “John is an assistant, associate, or full professor”. ’ In this paper, I argue that these characterizations of IIs cannot be right. Describing IIs in this way is misleading, because it seems that the IIs of SMs should be about *any number larger than three* in (3a), or about a disjunction of every rank above *assistant professor* in (3b). In particular, I argue that the IIs we attribute to SMs should not be characterized as a list of disjunctive statements, but as a disjunction of the form advocated by Büring (2007), and defended by Schwarz (2013):

(4) a. $[\![(3a)]\!] \rightsquigarrow$ *the speaker doesn’t know whether exactly three or more than three boys left*

b. $[\![(3b)]\!] \rightsquigarrow$ *the speaker doesn’t know whether John is an assistant professor or some higher rank*

Schematically, we can summarize both views as follows –for (3a):

(5) a. [*assistant professor*] or [*associate professor*] or [*full professor*]

b. [*assistant professor*] or [*more than an assistant professor*]

One may complain that, at first glance, the difference between (5a) and (5b) is trivial. I show that, if we take the form of these two formulations seriously, they make different predictions about the kind of inferences that are available about what the speaker is ignorant about, and hence about what the amount of information that is compatible with an utterance containing an SM is. Formulations like (5a) predict that every single disjunct is an epistemic possibility compatible with all the speaker knows (in this case, the speaker is predicted to believe that every rank above *assistant professor* is a possible rank for Bill). However, this characterization is not empirically adequate, since the speaker could felicitously utter (3b) even if she knew that Bill cannot be an associate professor. It is useful, then, to be careful about the distinction between what the exact IIs of SMs are, on the one hand, and what kind of situations they are compatible with, on the other.

Thus, the first task is to clarify what the exact form of the IIs that come with SMs are, and what they tell us about the speaker’s epistemic state after she uttered a sentence containing an

SM. In order to do so, I use the behavior of disjunction and its IIs as a benchmark to analyze the IIs of SMs. The next step is to show how we can better account for these IIs. In this paper I will focus exclusively on Superlative Modified Numerals (SMNs henceforth), and so I refer the reader to Mendia (2015) for discussion of a possible way to extend the analysis to other cases.

2. *What it means to be ignorant about something*

To properly talk about IIs, we first need to know what it means to be ignorant about something. A good starting point is to look at the behavior of disjunctions and move later to the connection with SMs. As it is well known, disjunctions can convey an II that the speaker does not know which of the disjuncts is true. Consider:

(6) Bill read Tintin or Asterix.

Upon hearing a sentence like (6), the addressee may draw a number of conclusions. Assuming the speaker is correct, she knows that Bill read Tintin or Asterix, which by itself is consistent with the possibility that Bill read both. Because the speaker does not specify which comic Bill read, the addressee may draw an inference that the speaker does not know which comic Bill read. This is precisely the type of IIs that this discussion will center on. Presumably, then, the existence of this kind of IIs is responsible for the oddness of texts where a disjunction is followed by a statement resolving the question as to what disjunct is true:

(7) Bill read Tintin or Asterix, #{namely/concretely} he read Asterix.

Let us be more specific about what it means to be ignorant. Assume that K and P stand for the familiar epistemic certainty and possibility operators, such that $\mathsf{K}_S\phi$ means *the speaker S knows that* ϕ and $\mathsf{P}_S\phi$ means that ϕ *is compatible with all S knows*. According to the properties that Hintikka (1962) ascribed to them, both operators K and P are interdefinable, since $\mathsf{K}\phi \leftrightarrow \neg\mathsf{P}\neg\phi$ and $\mathsf{P}\phi \leftrightarrow \neg\mathsf{K}\neg\phi$.

For concreteness, I follow Hintikka's (1962) epistemic logic, also used by Gazdar (1979). The semantics of this logic are defined by a pair $\langle\mathscr{W},\mathscr{R}\rangle$, a *frame*, where $\mathscr{W}$ is a non–empty set of possible worlds and $\mathscr{R}$ is an accessibility relation between worlds, such that for $w_0, w_1 \in \mathscr{W}$, $w_0\mathscr{R}w_1$ means that the world w_1 is accessible from w_0 (i.e., that the truths of w_1 are live possibilities for w_0). Frames are used to construct models by defining an interpretation function v such that, for every world $w \in \mathscr{W}$, v specifies the truth value of all propositions in w. The accessibility relation $\mathscr{R}$ is reflexive (8f) and transitive (8g). A model $\mathscr{M}$ is then a triple $\langle\mathscr{W},\mathscr{R},v\rangle$, where the truth of a formula with respect to a model $\mathscr{M}$ and a world w is recursively defined in the usual way, in (8a)–(8e). The logic described here corresponds to the KT4 system, which enriches the propositional calculus described above with the set of axioms in (8h)–(8j).[1]

(8) a. For some proposition p, if $v(w,p)$ in $\mathscr{M}$ then $\mathscr{M},w \models p$

b. $\mathscr{M},w \models \neg p$ *iff* $\mathscr{M},w \not\models p$

c. $\mathscr{M},w \models (p \wedge q)$ *iff* $\mathscr{M},w \models p$ and $\mathscr{M},w \models q$

[1]Whether KT4 is the most adequate logic to model knowledge and belief is a matter subject to philosophical debate; it is the system that Hintikka (1962) settled for, but see Hendricks and Symons (2014) for discussion.

d. $\mathscr{M}, w \models \mathsf{K}p$ *iff* for every $w' \in \mathscr{W}$, if $w\mathscr{R}w'$ then $\mathscr{M}, w' \models p$

e. $\mathscr{M}, w \models \mathsf{P}p$ *iff* for some element $w' \in \mathscr{W}$, $w\mathscr{R}w'$ and $\mathscr{M}, w' \models p$

f. $\mathscr{R}$ is REFLEXIVE *iff* for all $w \in \mathscr{W}, w\mathscr{R}w$

g. $\mathscr{R}$ is TRANSITIVE *iff* $(w\mathscr{R}w' \wedge w'\mathscr{R}w'') \rightarrow w\mathscr{R}w''$, for all $w, w', w'' \in \mathscr{W}$

h. Axiom K: $\mathsf{K}(p \rightarrow q) \rightarrow (\mathsf{K}p \rightarrow \mathsf{K}q)$ [Distribution Axiom]

i. Axiom T: $\mathsf{K}p \rightarrow p$ [Reflexivity Axiom]

j. Axiom 4: $\mathsf{K}p \rightarrow \mathsf{KK}p$ [Positive Introspection]

In this system, the following equivalences follow: $\mathsf{K}\neg\phi \equiv \neg\mathsf{P}\phi$ and $\neg\mathsf{K}\phi \equiv \mathsf{P}\neg\phi$. Then, to be ignorant about a proposition ϕ is expressed as follows:[2]

(9) $\neg\mathsf{K}[\phi] \wedge \neg\mathsf{K}\neg[\phi] \;\equiv\; \mathsf{P}[\phi] \wedge \mathsf{P}\neg[\phi]$

(9) shows the technical notion of *ignorance* that I shall refer to. I will take it that to be ignorant about ϕ is a stronger notion than the mere lack of knowledge about ϕ. By *being ignorant about* ϕ I refer to a mental (epistemic) state of some agent in which she is unsure about the truth of ϕ. Crucially, in order to be ignorant about ϕ it is necessary that the agent consider both ϕ and $\neg\phi$ live possibilities compatible with her knowledge. It follows that not only does the agent not know the truth of ϕ, she also does not know the truth of $\neg\phi$. Hintikka (1962:12–15) illustrates this difference by alluding to the contrast between *knowing that* ϕ and *knowing whether* ϕ:

(10) a. The speaker S does not know that ϕ: $\neg\mathsf{K}_S\phi$

b. The speaker S does not know whether ϕ: $\neg\mathsf{K}_S\neg\phi \wedge \neg\mathsf{K}_S\phi$

The distinction between (10a) and (10b) is in accordance with the intuition that when we are ignorant about whether ϕ, we consider both ϕ and $\neg\phi$ to be epistemic possibilities; I take this for granted in this paper. Sometimes I will use the following notational convention, where $\mathsf{I}_S[\phi]$ means that *the speaker is ignorant about whether* ϕ:[3]

(11) a. $\mathsf{I}_S[\phi] \equiv \neg\mathsf{K}_S[\phi] \wedge \neg\mathsf{K}_S\neg[\phi]$

b. $\mathsf{I}_S[\phi] \equiv \mathsf{P}_S[\phi] \wedge \mathsf{P}_S\neg[\phi]$

We turn now to the question of how to derive IIs of this form for disjunctive statements. Gazdar (1979), putting together insights from both Hintikka's (1962) epistemic logic and H.P. Grice's theory of language use (see Grice 1989), argued that they can be derived as clausal quantity implicatures. The formal principles responsible for IIs that I present in this paper also rely on Hintikka's epistemic logic and on Gricean reasoning, but the implementation will be a different one.[4] Assume, then, that we are dealing with a cooperative speaker and that some version of the Maxims of Quality are at work (Grice 1989).

[2] I use square brackets '[]' to enclose propositions, so that ϕ is a variable (that may itself stand for a proposition), and $[\phi]$ is *the proposition that* ϕ.

[3] I will use $\mathsf{P}[\phi]$ and $\neg\mathsf{K}\neg[\phi]$ interchangeably, as well as $\neg\mathsf{K}[\phi]$ and $\mathsf{P}\neg[\phi]$, the choice depending on what expression is more intuitive on a case to case basis.

[4] Gazdar (1979:59) derives IIs by applying a function f such that, for any two propositions ψ and ϕ, $f([\phi]) = \{x : x \in \{\mathsf{P}[\psi], \mathsf{P}\neg[\psi]\}\}$, such that $[\phi] \rightsquigarrow \mathsf{I}[\psi])$ *iff* (i) $[\psi] \rightarrow [\phi]$, (ii) $[\phi] \nrightarrow [\psi]$ and (iii) $[\phi] \nrightarrow \neg[\psi]$). Thus, a sentence $[\phi]$ implicates ignorance about whether $[\psi]$, only if $[\psi]$ entails $[\phi]$, but neither $[\psi]$ nor $\neg[\psi]$ is entailed by $[\phi]$. All Gazdar (1979) needs to derive an II is a relation of asymmetric entailment between two propositions. The method presented in this paper is no different in this sense, but it will benefit from the more general neo–Gricean framework.

(12) MAXIMS OF QUALITY

a. Do not say what you believe to be false.

b. Do not say what you do not have evidence for.

The Maxims of Quality can be related to the operators K and P by Hintikka's (1962:79) principle of EPISTEMIC IMPLICATION, whereby utterance of a sentence ϕ by a speaker S commits S to the knowledge of ϕ: ϕ implicates ψ if $\mathsf{K}[\phi \wedge \neg\psi]$ is inconsistent.[5] When a cooperative speaker S is following the Maxims of Quality, the addressee is allowed to infer that the utterance of ϕ by S implicates that $\mathsf{K}_S\phi$. This inference is sometimes also referred to as a Quality Implicature. In order to derive IIs, however, we need some notion of *strength*. For concreteness, assume the following characterization of the Maxim of Quantity.

(13) MAXIM OF QUANTITY
If two propositions $[\phi]$ and $[\psi]$ are such that (i) the denotation of $[\phi]$ asymmetrically entails $[\psi]$ (i.e., $[\phi] \rightarrow [\psi] \wedge \neg([\psi] \rightarrow [\phi])$), (ii) $[\phi]$ and $[\psi]$ are relevant, and (iii) the speaker believes both $[\phi]$ and $[\psi]$ to be true, the speaker should choose $[\phi]$ over $[\psi]$.

The Maxim of Quantity ensures that, given a number of true and relevant alternatives to the proposition that has been uttered, if a speaker is being cooperative, she should choose the semantically strongest alternative over the rest. In view of this definition of the Maxim of Quantity, it is useful to define the notion of Stronger Alternative (*SA*): An *SA* ψ of a proposition ϕ is an alternative proposition that asymmetrically entails ϕ: ψ is an *SA* of ϕ *iff* $\psi \rightarrow \phi$ and $\phi \nrightarrow \psi$. The set of *SA*s of a proposition ϕ is expressed as $SA(\phi)$. According to the Maxim of Quantity, if we are to be cooperative, we have to provide the semantically strongest relevant and true proposition we can. Following the terminology in Sauerland (2004), we now define the weakest form of inference, a Primary Implicature. In addition, we also define the Implicature Base, the set of propositions resulting from conjoining the Quality Implicature with its Primary Implicatures (Schwarz 2013).

(14) PRIMARY IMPLICATURE:
The inference that $\neg\mathsf{K}\psi$, for an *SA* ψ.

(15) IMPLICATURE BASE:
The set consisting of the Quality Implicature together with all its Primary Implicatures.

The motivation for drawing a Primary Implicature is provided by Gricean reasoning. Consider (6) again: *Bill read Tintin or Asterix*. As I mentioned above, (6) conveys the II that the speaker does not know which of the comics Bill read. The reasoning proceeds as follows: Assume that the speaker is being cooperative. This means that she is observing the Maxims of Quality and the Maxim of Quantity. Upon hearing (6) (represented as $[\mathsf{T} \vee \mathsf{A}]$), the addressee can conclude, then, that the speaker thinks that this much is true. Thus, by the principle of EPISTEMIC IMPLICATION, she concludes that $\mathsf{K}_S[\mathsf{T} \vee \mathsf{A}]$. $[\mathsf{T} \vee \mathsf{A}]$ has at least two stronger alternatives, the individual disjuncts $[\mathsf{T}]$ and $[\mathsf{A}]$. This follows from the Maxim of Quantity: $[\mathsf{T}] \in SA([\mathsf{T} \vee \mathsf{A}])$, since $[\mathsf{T}]$ is relevant and $[\mathsf{T}] \rightarrow [\mathsf{T} \vee \mathsf{A}]$, but $[\mathsf{T} \vee \mathsf{A}] \nrightarrow [\mathsf{T}]$.[6] The same reasoning applies also to $[\mathsf{A}]$. Following

[5] This notion of 'implication' is closer to that of 'entailment' and it is not the one that I will use when talking about implications in general.

[6] I will gloss over what happens when we factor in *SA*s that are formed by means of substituting scalar items that belong to the same Horn–Set. In the case at hand, the *SA* $[\mathsf{T} \wedge \mathsf{A}]$ does not play a role in deriving the IIs about each particular disjunct.

the Maxim of Quantity, the addressee concludes that if the speaker did not utter any of the *SA*s, it must be because she did not have evidence enough, or maybe she did not know. Therefore, she infers the Primary Implicatures that $\neg \mathsf{K}_S[\mathsf{T}]$ and $\neg \mathsf{K}_S[\mathsf{A}]$. (16) below summarizes the relevant propositions:

(16) a. ASSERTION: $[(6)] = [\mathsf{T} \vee \mathsf{A}]$

b. EPISTEMIC IMPLICATION: $\mathsf{K}_S[\mathsf{T} \vee \mathsf{A}]$

c. $SA([\mathsf{T} \vee \mathsf{A}]) = \{[\mathsf{T}], [\mathsf{A}]\}$

d. PRIMARY IMPLICATURES: $\neg \mathsf{K}_S[\mathsf{T}] \wedge \neg \mathsf{K}_S[\mathsf{A}]$

e. IMPLICATURE BASE: $\mathsf{K}_S[\mathsf{T} \vee \mathsf{A}] \wedge \neg \mathsf{K}_S[\mathsf{T}] \wedge \neg \mathsf{K}_S[\mathsf{A}]$

The Implicature Base contains all the information that the addressee may be able to deduce from the speakers utterance without any further assumptions. In particular, according to (16e), the addressee can conclude that the speaker knows that Bill read either Tintin or Asterix, that it is not the case that she knows that Bill read Tintin and it is not the case that she knows that Bill read Asterix. These are not quite yet the IIs we want. The last step to derive the right IIs from (16e) involves deriving that each disjunct is an epistemic possibility by the speaker, i.e., $\mathsf{P}_S[\mathsf{T}]$ and $\mathsf{P}_S[\mathsf{A}]$.[7] Luckily, the task is trivial: Given the properties of the operators K and P defined above, $\mathsf{P}_S[\mathsf{T}]$ and $\mathsf{P}_S[\mathsf{A}]$ are in fact entailed by the Implicature Base.

(17) a. $\mathsf{K}[\mathsf{T} \vee \mathsf{A}] \wedge \neg \mathsf{K}[\mathsf{T}] \wedge \neg \mathsf{K}[\mathsf{A}] \rightarrow \mathsf{P}[\mathsf{T}] \wedge \mathsf{P}[\mathsf{A}]$

b. *Proof*. Assume that $\neg \mathsf{P}[\mathsf{T}]$. Since $\mathsf{P}[\mathsf{T}]$ is equivalent to $\neg \mathsf{K} \neg [\mathsf{T}]$, then $\neg \mathsf{P}[\mathsf{T}]$ is equivalent to $\neg\neg \mathsf{K} \neg [\mathsf{T}]$, which can be reduced to $\mathsf{K} \neg [\mathsf{T}]$ by double negative. But $\mathsf{K} \neg [\mathsf{T}]$ cannot be, since it contradicts the Primary Implicature in the premise. Thus, it must be the case that $\neg \mathsf{K} \neg [\mathsf{T}]$, which is equivalent to $\mathsf{P}[\mathsf{T}]$. (The same proof holds *mutatis mutandi* for $\mathsf{P}[\mathsf{A}]$.)

The Implicature Base alone provides all the necessary pieces to derive that the epistemic possibility of every disjunct is a must.[8] It follows, too, that knowledge about the truth of any of the particular disjuncts should not be allowed, as we saw above in (7). In this case, this happens because both $\mathsf{K}_S[\mathsf{T}]$ and $\mathsf{K}_S \neg [\mathsf{T}]$ contradict the II that $\mathsf{I}_S[\mathsf{T}]$ –similarly for $\mathsf{K}_S[\mathsf{A}]$ and $\mathsf{K}_S \neg [\mathsf{A}]$.

The upshot of this discussion is that the choice of what counts as an *SA* is important: Given the right choice of *SA*s, IIs may be entailed by the Implicature Base. This is especially relevant when we consider disjunctions with multiple disjuncts.

(18) Bill read Tintin, Asterix or Conan.

Like (6), (18) may also convey that the speaker is uncertain as to which one of the three comics Bill read. What is important is that, just as before, all three comics must be considered epistemic possibilities by the speaker, otherwise the sentence is odd, as the following examples show.

[7]Recall that $\mathsf{P}_S[\mathsf{T}]$ is equivalent to $\neg \mathsf{K}_S \neg [\mathsf{T}]$ and the latter, together with the Primary Implicature $\neg \mathsf{K}_S[\mathsf{T}]$ constitutes the II that we are after, $\neg \mathsf{K}_S \neg [\mathsf{T}] \wedge \neg \mathsf{K}_S[\mathsf{T}]$ or $\mathsf{I}_S[\mathsf{T}]$.

[8]This is why I ignored the *SA* $[\mathsf{T} \wedge \mathsf{A}]$ above: Because even after adding a Primary Implicature like $\neg \mathsf{K}_S[\mathsf{T} \wedge \mathsf{A}]$, the Implicature Base in (16e) does not entail that $\mathsf{P}_S[\mathsf{T} \wedge \mathsf{A}]$, and so no II can be derived about $[\mathsf{T} \wedge \mathsf{A}]$. This may not be a bad thing; implicatures associated with the conjunctive alternative to disjunctive statements can sometimes be strengthened to $\mathsf{K}_S \neg [\mathsf{T} \wedge \mathsf{A}]$ and so constitute a Secondary Implicature (or Scalar Implicature), which are out of the scope of this paper. This is not to say that (6) is incompatible with the speaker's ignorance as to whether Bill read both comics. However, it should be clear that the model developed in this paper predicts that the addressee cannot draw an II about whether $\mathsf{K}_S[\mathsf{T} \wedge \mathsf{A}]$.

(19) Bill read Tintin, Asterix or Conan, #but he didn't read {Tintin/Asterix/Conan}.

Suppose that we want to derive the IIs conveyed by (19) exactly as before. The addressee would have to conclude first that the speaker is being cooperative, and so her utterance carries the EPISTEMIC IMPLICATION that $\mathsf{K}_S[\mathsf{T} \vee \mathsf{A} \vee \mathsf{C}]$. Since each individual disjunct asymmetrically entails the assertion, she derives the Primary Implicature that $\neg\mathsf{K}_S[\phi]$ for every $[\phi] \in SA([\mathsf{T} \vee \mathsf{A} \vee \mathsf{C}])$ which, together with the assertion, forms the Implicature Base.

(20) a. ASSERTION: $[(18)] = [\mathsf{T} \vee \mathsf{A} \vee \mathsf{C}]$

b. EPISTEMIC IMPLICATION: $\mathsf{K}[\mathsf{T} \vee \mathsf{A} \vee \mathsf{C}]$

c. $SA([\mathsf{T} \vee \mathsf{A} \vee \mathsf{C}]) = \{[\mathsf{T}], [\mathsf{A}], [\mathsf{C}]\}$

d. PRIMARY IMPLICATURES: $\neg\mathsf{K}[\mathsf{T}] \wedge \neg\mathsf{K}[\mathsf{A}] \wedge \neg\mathsf{K}[\mathsf{C}]$

e. IMPLICATURE BASE: $\mathsf{K}[\mathsf{T} \vee \mathsf{A} \vee \mathsf{C}] \wedge \neg\mathsf{K}[\mathsf{T}] \wedge \neg\mathsf{K}[\mathsf{A}] \wedge \neg\mathsf{K}[\mathsf{C}]$

As before, the only *SA*s that the system has access to are the individual disjuncts $[\mathsf{T}], [\mathsf{A}]$ and $[\mathsf{C}]$. However, in this case, the Implicature Base in (20e) does not entail the right kind of IIs. In particular, (20e) does not entail that every individual disjunct is an epistemic possibility for the speaker: $\mathsf{P}[\phi]$, for all $[\phi] \in SA([\mathsf{T} \vee \mathsf{A} \vee \mathsf{C}])$. The problem is that unlike in the case of (6) above the entailment pattern [Implicature Base $\rightarrow$ IIs] is not a logical truth. This can be demonstrated by constructing a model where taking the Implicature Base as a premise, the epistemic possibility of some individual disjunct does not follow. (21) below provides such a counter–model for $\mathsf{P}[\mathsf{A}]$: In (21) the Implicature Base (20e) is true in w_0, but it is not the case that $\mathsf{P}[\mathsf{A}]$ in w_0. Similar models can be constructed to show that neither $\mathsf{P}[\mathsf{A}]$ nor $\mathsf{P}[\mathsf{C}]$ are logical consequences of the Implicature Base in (20e).

(21) Counter–model for $\mathsf{K}[\mathsf{T} \vee \mathsf{A} \vee \mathsf{C}] \wedge \neg\mathsf{K}[\mathsf{T}] \wedge \neg\mathsf{K}[\mathsf{A}] \wedge \neg\mathsf{K}[\mathsf{C}] \rightarrow \mathsf{P}[\mathsf{A}]$ (where $w_0 \mathscr{R} w_1, w_2, w_3$)

w_0 : $[\mathsf{T}], \neg[\mathsf{A}], \neg[\mathsf{C}]$
w_1 : $[\mathsf{C}], \neg[\mathsf{A}], \neg[\mathsf{T}]$
w_2 : $[\mathsf{T}], \neg[\mathsf{A}], \neg[\mathsf{C}]$
w_3 : $[\mathsf{T}], \neg[\mathsf{A}], \neg[\mathsf{C}]$

As shown by Alonso-Ovalle (2006), a solution to this situation can be provided by including 'sub-domain' alternatives. In the case of disjunction, sub–domain alternatives are alternatives formed by smaller disjunctions each of whose individual disjuncts are part of the assertion (see also Chierchia 2013). Given the definition of *SA*s in terms of asymmetric entailment, sub–domain alternatives of multiple disjuncts constitute all *SA*s. In the case of (18), the revised set of *SA*s, Primary Implicatures and Implicature Base is as in (22):

(22) a. $SA([\mathsf{T} \vee \mathsf{A} \vee \mathsf{C}]) = \{[\mathsf{T} \vee \mathsf{A}], [\mathsf{T} \vee \mathsf{C}], [\mathsf{A} \vee \mathsf{C}], [\mathsf{T}], [\mathsf{A}], [\mathsf{C}]\}$

b. PRIMARY IMPLICATURES:
$\neg\mathsf{K}[\mathsf{T} \vee \mathsf{A}] \wedge \neg\mathsf{K}[\mathsf{T} \vee \mathsf{C}] \wedge \neg\mathsf{K}[\mathsf{A} \vee \mathsf{C}] \wedge \neg\mathsf{K}[\mathsf{T}] \wedge \neg\mathsf{K}[\mathsf{A}] \wedge \neg\mathsf{K}[\mathsf{C}]$

c. IMPLICATURE BASE:
$\mathsf{K}[\mathsf{T} \vee \mathsf{A} \vee \mathsf{C}] \wedge \neg\mathsf{K}[\mathsf{T} \vee \mathsf{A}] \wedge \neg\mathsf{K}[\mathsf{T} \vee \mathsf{C}] \wedge \neg\mathsf{K}[\mathsf{A} \vee \mathsf{C}] \wedge \neg\mathsf{K}[\mathsf{T}] \wedge \neg\mathsf{K}[\mathsf{A}] \wedge \neg\mathsf{K}[\mathsf{C}]$

All the shorter disjuncts can be derived as Primary Implicatures, since for any two propositions $[d_1 \vee d_2]$ and $[d_1 \vee d_2 \vee d_3]$, $[d_1 \vee d_2] \rightarrow [d_1 \vee d_2 \vee d_3]$, but $[d_1 \vee d_2 \vee d_3] \nrightarrow [d_1 \vee d_2]$. With the addition of these sub–domain *SA*s, the revised Implicature Base in (22c) entails that every individual disjunct is an epistemic possibility.

(23) a.
$$\left.\begin{array}{ll} \mathsf{K}[\mathsf{T} \vee \mathsf{A} \vee \mathsf{C}] & \wedge \\ \neg\mathsf{K}[\mathsf{T} \vee \mathsf{A}] & \wedge \\ \neg\mathsf{K}[\mathsf{T} \vee \mathsf{C}] & \wedge \\ \neg\mathsf{K}[\mathsf{A} \vee \mathsf{C}] & \wedge \\ \neg\mathsf{K}[\mathsf{T}] \wedge \neg\mathsf{K}[\mathsf{A}] \wedge \neg\mathsf{K}[\mathsf{C}] & \end{array}\right\} \rightarrow \mathsf{P}[\mathsf{T}] \wedge \mathsf{P}[\mathsf{A}] \wedge \mathsf{P}[\mathsf{C}]$$

b. *Proof*. Assume that $\neg\mathsf{P}[\mathsf{T}]$. This is equivalent to $\mathsf{K}\neg[\mathsf{T}]$. Together with the EPISTEMIC IMPLICATION of the assertion, we have that $\mathsf{K}[\mathsf{T} \vee \mathsf{A} \vee \mathsf{C}] \wedge \mathsf{K}\neg[\mathsf{T}] \rightarrow \mathsf{K}[\mathsf{A} \vee \mathsf{C}]$. However, the consequence $\mathsf{K}[\mathsf{A} \vee \mathsf{C}]$ contradicts the premise that $\neg\mathsf{K}[\mathsf{A} \vee \mathsf{C}]$, and so it cannot be the case that $\neg\mathsf{P}[\mathsf{T}]$. (Similar for $[\mathsf{A}]$ and $[\mathsf{C}]$.)

Similar proofs can be constructed for disjunctions with more than three disjuncts. Thus, IIs of disjunctive statements can be derived by relying on independently needed formal principles, which provide the two necessary —and sufficient— ingredients to derive IIs about each particular disjunct: A suitable epistemic logic and the assumption that *SA*s are established by asymmetric entailment relations.[9] The question now is whether the same system can be extended to SMs. Before turning into *how* to derive the IIs of SMs, however, we first need to decide *what* exactly is the form of the IIs that come with SMs.

3. *Characterizing ignorance with SMNs*

The last section showed the exact nature of IIs introduced by disjunction together with one way to derive them. The next step is to assess to which extent the IIs that have been associated with SMs resemble the IIs associated with disjunctions. As mentioned above, the focus will be solely on the case of Superlative Modified Numerals, or SMNs for short.

(24) a. At least four friends came to the party.

b. At most four friends went to swim.

Upon hearing the sentences in (24), an addressee may infer that the speaker does not know exactly how many friends were involved in either activity. This uncertainty can be understood in a variety of ways. As advanced in §1, one can find two main views in the literature about what exactly the IIs of a sentence like (24a) are, summarized below:

(25) a. OPTION 1: For any number n such that $n \geq 4$, the speaker is ignorant about whether or not exactly n friends came to the party.

b. OPTION 2: The speaker is ignorant about whether or not exactly 4 or more than 4 friends came to the party.

[9]This contrast with Gazdar's (1979) way of deriving IIs. In his system, IIs are derived by means of a function that computes clausal quantity implicatures, that only applies to a specific set of propositions that meet a particular criteria (see foonote 2). His system aims to cover a broader range of data than just IIs with disjunction, and so this is not the place for a thorough comparison. But it is worth noting that, under the light of disjunction alone, there is no need to resort to a more complex mechanism.

IIs like (25a) are explicitly mentioned by Mayr (2013:158),[10] whereas IIs like (24b) were first argued for Büring (2007). Both approaches make different predictions. For instance, assuming a reduced domain of 8 friends, an II like (25a) explicitly states that the speaker is ignorant about whether exactly four, five, six, seven or eight friends came to the party. What OPTION 1 is saying is that the addressee is entitled to infer that the speaker does not know whether exactly six friends came to the party, for instance. This is not so in the case of OPTION 2: All the addressee can infer from the speaker's statement in (25b) is that she knows that either exactly four or more than four friends came; but nothing more can be inferred —i.e., there cannot be any specific inference about whether exactly six friends came to the party.

The task now is to decide which one of OPTION 1 and OPTION 2 better characterizes the IIs that come with SMs. The rest of this section is devoted to show that OPTION 2 is more adequate.

3.1. Partial ignorance

The first piece of evidence that IIs of SMs are like by OPTION 2 is that sentences containing SMs need not convey ignorance with respect to any a number above (for *at least*) or below (for *at most*) the one that is mentioned. Consider:

(26) **Situation:** *Two commentators are talking on TV about a classic basketball game of the 90's. They are commenting on the amount of points that were scored in that game on triples. A commentator says: <u>Michael Jordan scored at least 30 points</u>.*

Both commentators know that triples are three–point field goals in basketball, in contrast to the two points awarded for easier shots. They assume, too, that they are targeting an audience that is well versed in the rules of basketball, and so this information is shared by every agent in the conversation, active or passive. In this situation, the commentator's utterance is perfectly acceptable. This is an instance of PARTIAL IGNORANCE: the addressee *A* cannot draw an inference that the speaker *S* is completely ignorant, since *A* knows that *S* does know something –namely, that quantities of scores that are not tuples of three are not allowable options. Similar examples can be constructed for *at most* too.

PARTIAL IGNORANCE speaks in favor of OPTION 2 over OPTION 1: The latter would predict an II such that for any number *n* above 30, the speaker is ignorant about whether Michael Jordan scored *n* points. But this is too strong an inference. OPTION 2, on the other hand, is compatible with those 'gaps' in the possibilities that the speaker is considering, and a knowledgeable enough addressee will not derive the II that the speaker is ignorant as to whether, e.g., Michael Jordan scored 31 points.[11]

Another manifestation of the same property can be traced down to examples where an SMN

[10] Geurts and Nouwen (2007:558) and Coppock and Brochhagen (2013b:10) also suggest that these are the right IIs for SMs that modify elements that participate in non–entailing scales. I will not consider these cases in this paper, and so I refer the reader to Mendia (2015) for further discussion.

[11] An anonymous reviewer suggests the following alternative to rescue OPTION 1: It could be that the points in the scale that stand in conflict with real–world knowledge can be rejected. The suggestion is an interesting one and deserves more attention than the little I can offer here. At first glance, however, a reason why I still would prefer OPTION 2 is the following. Suppose that if a Primary Implicature contradicts some statement in the Common Ground, it is automatically rejected. If that were the case we would need a theory of *when* exactly Common Ground information can override a Primary Implicature. For instance, we would have to know why IIs of disjunction can never be overriden by the Common Ground (see §2 above), and why secondary (scalar) implicatures can never be overriden (see Magri 2009). By contrast, no such explanation is necessary if we accept that SMs convey PARTIAL IGNORANCE.

is used in situations where the bounds denoted by the SMN are not fixed. This is to say that *at most* is compatible with a flexible lower bound, and *at least* is compatible with a flexible upper bound. Consider the following example (from Nouwen 2015):

(27) **Situation:** *Bill does not remember the password of his WIFI network. The only thing he remembers is that the password is between six and ten characters long.*

a. The password is at least six characters long.

b. The password is at most ten characters long.

In the situation above, Bill can utter both (27a) and (27b) felicitously, even though his epistemic state excludes some of the values that in principle could be available. Moreover, the speaker can utter both sentences without bringing attention to any misleading implicatures. To this respect, SMNs differ radically from disjunction in their behavior. As discussed earlier, disjunction always requires TOTAL IGNORANCE: In a disjunctive statement, the addressee draws the inference that, for all the speaker knows, every disjunct constitutes an epistemic possibility. That is, upon hearing a sentence like (18) above –repeated below–, the addressee will invariably draw the inference that the speaker does not know exactly what comic Bill read.

(18) Bill read Tintin, Asterix or Conan.

This suggests that there is something fundamentally different between SMs and disjunctions. Both constructions allow for IIs, albeit of a different nature: Only SMNs are compatible with some knowledge (hence the term PARTIAL IGNORANCE). In (18), the speaker mentioned three options that are in accordance with her knowledge. Due to the assumption that the speaker is cooperative, and therefore that she is following the Maxim of Quantity, the addressee assumes that there is no true stronger statement that she could have uttered instead of (18). Here we could also invoke the third statement of the Maxim of Manner, BREVITY: *Be brief, avoid unnecessary prolixity*. If the addressee assumes that the speaker is being cooperative and following the Maxims of Quality, Quantity and Manner, there is no reason she should have mentioned three disjuncts in (18) if she knew that any one of them was not a plausible option.

3.2. Assertibility vs. verifiability

It is important to be careful about the distinction between (i) what amount of speaker knowledge SMNs require so that they can be used felicitously and (ii) what kind of IIs they are compatible with. The first correspond to the *assertibility* conditions, or to what a speaker *must* know and what the speaker *must not* know in order to use an SMN. The second correspond to the *verifiability* conditions of sentences with SMNs, or what kind of information (scenarios) are compatible with the use of SMNs.

As before, it is useful to compare the behavior of SMNs with disjunction. Suppose, following Hintikka (1962) and Gazdar (1979) among others, that a cooperative speaker will felicitously utter a proposition ϕ if, minimally, she knows/believes that the assertion conveyed by ϕ is true, or she has enough grounds to believe so. Suppose that ϕ is of the form $[\psi \vee \chi]$. In that case, what does it mean to know or to have evidence enough for $[\psi \vee \chi]$? The minimal assertibility conditions for disjunction seem to be essentially modal: They require that the speaker considers that both ψ and χ are possibly (but not certainly) true. Zimmermann (2000), following this lead, proposed that disjunctive sentences are interpreted as a conjunction of different epistemic possibilities, such that $[\psi \vee \chi]$ is interpreted as $\Diamond[\psi] \wedge \Diamond[\chi]$. Thus, asserting ϕ requires that ϕ is true, whereas asserting $[\psi \vee \chi]$ requires both that $[\psi \vee \chi]$ is true and that each of $[\psi]$ and $[\chi]$ is possibly true. This intuition seems to be essentially correct, as suggested by (28).

(28) Bill ate an apple or a pear, #but I know that he didn't eat an apple.

Surely, (28) is not a logical contradiction, but it is still not felicitous. The same contrast is found with SMNs too.

(29) a. Bill ate at least two apples, #but I know that he didn't eat two.
 b. Bill ate at most two apples, #but I know that he didn't eat two.

The standard analysis of SMNs in terms of Generalized Quantifiers predicts that (29a) should be truth-conditionally equivalent to the same sentence with *more than one*. This analysis is agnostic as to what the assertibility conditions of SMNs are, and whether they are the same of equivalent expressions, like in the case of *more than one*. It is obvious, however, that both constructions are not assertible in the same set of circumstances, since *Bill ate more than one apple* is compatible with a follow–up like *but I know that he didn't eat two*.[12]

What (29a) suggests is that part of what the speaker must know in order to felicitously utter a sentence with an SMN is that the number that is mentioned and that the SMN is modifying must be a possibility. However, this is not enough as a description of the assertibility conditions of SMNs. Consider:

(30) Bill ate at least two apples…
 a. #but I know that he didn't eat more than two.
 b. #and there is no number above two such that he could have eaten that number of apples.
 c. but I know that he didn't eat {three/four/three or four/between three or six/…}.

(31) Bill ate at most six apples…
 a. #but I know that he didn't eat fewer than six.
 b. #and there is no number below six such that he could have eaten that number of apples.
 c. but I know that he didn't eat {five/four/three or four/between two or five/…}.

The examples in (30c)/(31c) are not surprising, given that SMNs convey only PARTIAL IGNORANCE, and so the speaker need not consider each larger or smaller number other than the one modified by the SMN to be a possibility. All these varied scenarios provide contexts that can verify an utterance with an SMN.

[12] The standard theory, as first presented in Barwise and Cooper (1981), captures the property of SMs that they denote lower (*at least*) and upper (*at most*) bounds: $[\![$*at most n*$]\!] = \lambda A.\lambda B\,.\,|A \cap B| \leq n$ and $[\![$*at least n*$]\!] = \lambda A.\lambda B\,.\,|A \cap B| \geq n$. A sentence of the form *at least n A's B* is false if less than *n A's B*. Similarly, a sentence of the form *at most n A's B* is false whenever more than *n A's B*.

The examples in (30a)/(30b) and (31a)/(31b) are the interesting ones. Just like (29), sentences with SMNs seem to be infelicitous if the speaker is certain that no value beyond/below the modified one is a possibility. For an example like *Bill ate at least two apples* this means that the speaker knows that some number of apples beyond two is a possibility, but she need not know anything else. She could, of course, as the continuations in (30c)/(30c) show, but all she *must* know is that there is some number above two for which it may be true that Bill ate that many apples. Again, these results favor OPTION 2 over OPTION 1 in (25) above: The IIs that cannot be challenged are precisely those advocated by OPTION 2.

What matters when uttering a sentence with an SMN is that there must be at least two possibilities that the speaker has to consider to be true: *exactly n* and *more than n* in the case of *at least n*, and *exactly n* and *fewer than n* for *at most n*. We can now summarize the assertibility conditions of SMNs as follows (cf. Cohen and Krifka 2014, Spychalska 2016):[13]

(32) ASSERTIBILITY CONDITIONS OF *at least n*: A proposition ϕ of the form AT LEAST *n As B* is assertible by a speaker *S iff*:

a. $[= n]$ is compatible with all *S* knows,

b. $[> n]$ is compatible with all *S* knows, and

c. it is not the case that $[< n]$.

(33) ASSERTIBILITY CONDITIONS OF *at most n*: A proposition ϕ of the form AT MOST *n As B* is assertible by a speaker *S iff*:

a. $[= n]$ is compatible with all *S* knows,

b. $[< n]$ is compatible with all *S* knows, and

c. it is not the case that $[> n]$.

To have explicit and well-defined assertibility conditions is useful to assess how well theories of SMNs behave with respect to the IIs they predict. Thus, the assertibility conditions in (32) and (33) can be straightforwardly represented by means of formulae in epistemic logic:

(34) a. $[\geq n]$ is assertible by a speaker *S iff* $\mathsf{K}_S[\geq n] \wedge \mathsf{P}_S[= n] \wedge \mathsf{P}_S[> n] \wedge \mathsf{K}_S\neg[< n]$

b. $[\leq n]$ is assertible by a speaker *S iff* $\mathsf{K}_S[\leq n] \wedge \mathsf{P}_S[= n] \wedge \mathsf{P}_S[< n] \wedge \mathsf{K}_S\neg[> n]$

These assertibility conditions correspond to what the epistemic state of a cooperative speaker has to be like so that a sentence with an SMN can be uttered felicitously. To this respect, the assertibility conditions of SMNs are fully parallel to those of disjunction, where each disjunct is required to be possibly true, and it is required not to be certainly true by the speaker. What the speaker knows for certain amounts to what sentences with SMNs assert: $\mathsf{K}_S\neg[< n]$ and $\mathsf{K}_S[\geq n]$ for *at least*, and $\mathsf{K}_S\neg[> n]$ and $\mathsf{K}_S[\leq n]$ for *at most*, according to the standard semantic analysis (Barwise and Cooper 1981; see footnote 12). The IIs that are derived correspond exactly to OPTION 2 in (25), as proposed by Büring (2007) for *at least*, and cannot be as in OPTION 1, as mentioned by Mayr (2013) and Coppock and Brochhagen (2013b), and also considered by Geurts and Nouwen (2007). A corollary of admitting that OPTION 2 is the right one is that IIs are derived by pragmatic means, just like those that come with disjunction.

[13] Here and throughout the paper I will use the following abbreviations: Numbers enclosed in square brackets '[ ]' stand for a proposition containing that number, such that $[= 2]$ stands for *Bill has exactly 2 children*. Similarly, I will use $[\leq 2]$, $[\geq 2]$, $[< 2]$ and $[> 2]$ for propositions containing the expressions *at most 2*, *at least 2*, *less than 2* and *more than 2*.

3.3. IIs as implicatures

A quick look at the basic tests for implicatures seems to support the view that IIs are implicatures: They are both cancelable and reinforceable.

(35) CANCELABILITY:
Bill has four kids. Yesterday he saw a sign at a supermarket: 'Huge sales and discounts for parents. To qualify, you must have at least three kids.' After reading it, Bill reasoned as follows: 'I qualify, <u>I have at least three kids. In fact, I have four</u>.'[14]

(36) REINFORCEABILITY:
Bill has at least three kids, but I have no idea how many exactly.

A further argument that IIs are implicatures is provided by an observation in Grice (1975), who pointed out –referring to disjunctive sentences– that IIs may be canceled when it is known by all the participants in the conversation that the speaker is not being maximally informative. These are cases where the speaker is not expected to provide all the relevant information that is available to her, in whichever form. The reason is the following: Usually, it is taken to be Common Ground that participants in a conversation obey the Maxim of Quantity. The Maxim of Quantity is a cooperative principle stating, roughly, that the speaker is expected to convey all the information she has available, that is, she is expected to provide the strongest relevant statement she is able to. IIs arise as a direct consequence of this mutual agreement. Put it otherwise: in the absence of the assumption that the speaker is following the Maxim of Quantity, speakers are not expected to be maximally informative, and so there could be stronger relevant propositions that they could have remained silent about, while still being cooperative. As it turns out, in situations where it is Common Ground that the speaker is not obeying the Maxim of Quantity, IIs are not present. Grice (1989:44–45) discusses the case of disjunction:

> 'I can say to my children at some stage in a treasure hunt, "The prize is either in the garden or in the attic. I know that because I know where I put it, but I'm not going to tell you." Or I could just say (in the same situation) "The prize is either in the garden or in the attic", and the situation would be sufficient to apprise the children the fact that my reason for accepting the disjunction is that I know a particular disjunct to be true.'

The treasure hunt scenario illustrates that the cancelation of the II is contingent upon knowing whether the different agents in the conversation have agreed on obeying the Maxim of Quantity or not. Consider now the following scenario (inspired by Fox 2014):

(37) **Situation:** *In a TV game show, utterances by the host are presupposed to disobey the Maxim of Quantity. The contestant has won the biggest prize, which consists of one of two options: She either takes \$5000 in cash or she takes an envelope with an amount of cash unknown to her, but that the audience and the host already know. The contestant has to gamble. At some point, the host decides to give a hint that will help the contestant*

[14] Same with disjunction: In some cases disjunction does not trigger an II, even in plain non–modal contexts. The following example is from Chierchia (2013), adapted from von Fintel and Gillies (2010): *We lost a ball. John is telling us that it is not in Box A. We saw it land in Box A or Box B; thus, the ball must be in Box B.* It seems that in the context above the inference that the speaker does not know what of the disjuncts is true has to be defeasible to make a coherent discourse.

to assess her chances of picking the most profitable choice. Of course, the hint is such that it only provides part of the information available to the host, and this is common understanding for both the contestant and the audience. In this case, the host says: The envelope contains at least \$3000.

Deactivating the Maxim of Quantity as in (37) makes the hint provided by the host appropriate even though he knows the exact quantity that the envelope contains. In addition, it precludes the contestant from drawing the inference that the host does not know how much money it contains. In fact, the contestant can be confident that the hint is true precisely because she takes the host to be an authority on the matter.

In sum, the fact that IIs depend on the shared knowledge that participants in the conversation are obeying the Maxim of Quantity strongly suggests that they are implicative in nature. This conclusion is further supported by the fact that they are cancelable and reinforceable.

4. Deriving ignorance with SMNs

This is where we stand: We have seen that IIs are pragmatic inferences that can be drawn from utterances containing SMNs, very much like disjunction. These IIs are better described in terms of a disjunctive statement of the form *exactly n or more/less than n* (which corresponds to OPTION 2 in (25) above). Like disjunction, both of these disjuncts are inferred to be epistemic possibilities for the speaker. But, unlike disjunction, drawing these inferences of SMNs does not result in TOTAL IGNORANCE, but in PARTIAL IGNORANCE.

And this is where we go: The next step is to explain how to derive the correct IIs. To do so, I first introduce Schwarz (2013, 2014) account which, together with the epistemic logic introduced above, derives the IIs of SMNs argued for in the previous section.

4.1. Taking stock

4.1.1. Büring's (2007) idea

The starting point is Büring (2007) who, building on Krifka (1999), proposed what many consider the best account to IIs of *at least* (although not for *at most*; we turn to this question later).[15] The general idea is to account of IIs with *at least* from general, neo–Gricean pragmatic principles, just as we did with disjunction above. In order to do so, Büring (2007) provided a syncategorematic definition of *at least*:

(38) *at least n A's B* $\stackrel{\text{def}}{=}$ *exactly n P Q* or *more than n P Q*

In (38) above ‘ $\stackrel{\text{def}}{=}$ ’ should be read *is interpreted as*. This definition captures directly the assertibility conditions of *at least* introduced in §3.2. The advantage of making this move is twofold: It not only captures the fact that the assertibility conditions of disjunctive sentences and *at least* are the same, but it allows to calculate implicatures following the same reasoning introduced above for disjunction:

[15] See Cummins and Katsos (2010), Biezma (2013), Schwarz (2013, 2014), Cohen and Krifka (2014), Penka (2014), Spychalska (2016) and Kennedy (2015), a.o.

(39) *Bill read at least two books*

a. ASSERTION: $[=2] \vee [>2]$

b. EPISTEMIC IMPLICATION: $\mathsf{K}_S([=2] \vee [>2])$

c. $SA([=2] \vee [>2]) = \{[=2], [>2]\}$

d. PRIMARY IMPLICATURES: $\neg\mathsf{K}_S[=2] \wedge \neg\mathsf{K}_S[>2]$

e. IMPLICATURE BASE: $\mathsf{K}_S([=2] \vee [>2]) \wedge \neg\mathsf{K}_S[=2] \wedge \neg\mathsf{K}_S[>2]$

Under the assumption that the speaker is cooperative, the fact that she did not utter any of the stronger alternatives prompts the addressee to conclude that the speaker does not have evidence or is uncertain about them. Moreover, it happens to be the case that the assertion together with the Primary Implicatures entail that the speaker considers each of the stronger alternatives a possibility, $\mathsf{P}_S[=2] \wedge \mathsf{P}_S[>2]$ (see the proof in (17) for details). Together with the Implicature Base, this provides amounts to the right kind of IIs: $\mathsf{I}_S[=2] \wedge \mathsf{I}_S[>2]$.

4.1.2. A conceptual worry

The main criticism of Büring's (2007) so–called disjunctive approach is purely conceptual. Even if we concede the descriptive adequacy of (38), how do we get from an SMN to a disjunctive statement? At what level is *at least n* equivalent to *exactly n or more than n*? The complaint in the literature is that the semantic equality between SMNs and the disjunction in (38) is unwarranted (see e.g., Geurts and Nouwen 2007, Coppock and Brochhagen 2013b), and, in fact, that is ultimately the reason why Büring's (2007) treatment is syncategorematic. The criticism goes as follows: The disjunctive operator *or* in (38) is part of the meta–language of the meaning of a proposition containing an SMN. Sentences containing SMNs are not disjunctions on the surface level, and nobody has suggested a syntactic transformation to turn SMNs in disjunctions at LF. Moreover, the only empirical motivation for such transformation is, to date, the fact that IIs can be drawn from SMNs, and in the absence of more evidence, the argument supporting the motivation for such operation would be circular. Thus, it is not likely that such complex transformation exists.

One may assume then that having *or* in the meta–language is sufficient to assure that SMNs will behave like disjunctions at some level. However, this reasoning does not come along either. As Coppock and Brochhagen (2013b) discuss, the fact that some proposition ϕ may be described as a disjunction in the meta–language does not seem to clarify the question as to why we can then draw implicatures from this disjunction. The reasoning would be like this: if a speaker S utters expression ϕ, such that the meta–language description of ϕ contains a disjunctive symbol, then S considers each disjunct of the meta–language description of ϕ to be possible. There is an obvious problem with this: the reasoning assumes speaker's knowledge (conscious of unconscious) of the meta–language description that semanticists use to describe the meaning of ϕ. But this is rather implausible, how could they know?

A second, maybe more natural option is to assume that SMNs are not disjunctive statements. But if we give up (38), we need to replace it with some other calculus that will successfully derive IIs of SMNs, without taking SMNs to be disjunctions at any level.

4.2. *Analysis*

4.2.1. *Choosing the right alternatives*

The upshot of the previous discussion is that a syncategorematic definition for *at least* lacks some explanatory grip. Schwarz (2013, 2014) provides a neo–Gricean account of IIs of *at least* that (i) aims to capture its right properties, and (ii) does so by operating within a standard set of neo–Gricean assumptions. His goal, ultimately, is to motivate the existence of a pair alternatives that will correspond to each of the disjuncts in Büring's (2007) definition of *at least* in (38). Such pair of alternatives would allow us to derive the correct IIs and to get rid of *at least*'s syncategorematic meaning. Schwarz's (2013) idea, also considered by Mayr (2013), is to consider more than one Horn–Set from which relevant propositional alternatives can be determined. Following the neo–Gricean practice, the set of alternative meanings $Alt(\phi)$ for an assertion ϕ is formed by the semantic meanings of syntactic structures obtained by substituting one or more scale–mates in the syntactic structure of ϕ with members of its Horn–Set (definition from Sauerland 2004):

(40) $Alt(\phi)$ = $\{\psi$: ψ is derivable from ϕ by successive replacement of scalar items with members of their Horn–Set$\}$

Schwarz leaves the alternative generation algorithm as in (40), and so $Alt(\phi)$ for any assertion ϕ is calculated by the usual Gricean substitution method. The following two relevant Horn–Sets in (41a) and (41b) are considered:

(41) a. Horn–Set$_1$: {*at least, exactly*}
b. Horn–Set$_2$: $\{1,2,3,4,...\}$

With these two Horn–Sets, the set of *SA*s in (42b) for an assertion (42a) is derived. As before, the Primary Implicatures (42c) can be drawn by negating the *SA*s under the assumption that the speaker is cooperative, and so she is following the Maxim of Quantity.

(42) a. ASSERTION: $[\geq 3]$ (e.g., *at least three boys left*)
b. $SA([\geq 3]) = \{[\geq 4], [\geq 5], [\geq 6], \ldots, [= 3], [= 4], [= 5], \ldots\}$
c. PRIMARY IMPLICATURES:
$\neg\mathsf{K}_S[\geq 4] \wedge \neg\mathsf{K}_S[\geq 5] \wedge \ldots \wedge \neg\mathsf{K}_S[= 3] \wedge \neg\mathsf{K}_S[= 4] \wedge \neg\mathsf{K}_S[= 5] \ldots$
d. IMPLICATURE BASE:
$\mathsf{K}_S[\geq 3] \wedge \neg\mathsf{K}_S[\geq 4] \wedge \neg\mathsf{K}_S[= i]$, for all $i > 2$

It suffices to include only the negation of the weakest Primary Implicature from the set of *SA*s in (42b) corresponding to the *at least* alternatives, $\neg\mathsf{K}_S[\geq 4]$ in this case. This is so because $\neg\mathsf{K}_S[\geq 4]$ is in fact stronger that the rest of *at least* alternatives since it entails $\neg\mathsf{K}_S[\geq i]$ for any $i > 4$. The *exactly* alternatives are non–monotonic, and so all of them must be factored in. Despite the size of the set of *SA*s, the Implicature Base in (42d) only entails that two of them are epistemic possibilities: $[= 3]$ and $[\geq 4]$. This can be shown to be a general consequence of the assumptions we have made about the epistemic properties of SMNs. We need to prove three things: (i) that (42d) entails $\mathsf{P}[= 3]$, (ii) that (42d) entails $\mathsf{P}[\geq 4]$, (iii) and that there is no n such that (42d) $\rightarrow \mathsf{P}[= n]$, for all $n > 3$. I will show that this is the case by proving that this particular example under discussion follows from the general case. First we check that (42d) entails $\mathsf{P}[=3]$.

(43) a. $\mathsf{K}[\geq i] \wedge \neg\mathsf{K}[\geq i+1] \wedge \neg\mathsf{K}[=i] \rightarrow \mathsf{P}[=i]$ for all $i \in \mathbb{N}$

b. *Proof*. Assume that $i=3$ and suppose that $\neg\mathsf{P}[=3]$. This is equivalent to $\mathsf{K}\neg[=3]$. Assuming that $\mathsf{K}\neg[=3]$ entails that $\mathsf{K}[\geq 4]$ is true, since otherwise the premise $\mathsf{K}[\geq 3]$ would be false: $\mathsf{K}\neg[=3] \wedge \mathsf{K}\neg[\geq 4] \wedge \mathsf{K}[\geq 3] \equiv \perp$. However, $\mathsf{K}[\geq 4]$ contradicts the premise that $\neg\mathsf{K}[\geq 4]$, and we conclude that $\neg\neg\mathsf{P}[=3]$, which is equivalent to $\mathsf{P}[=3]$.

Thus, when uttering a sentence of the form *SMN n P Q*, the addressee will invariably infer that *exactly n P Q* is an epistemic possibility for the speaker. The reason is that $\mathsf{P}[=i]$ is logically entailed by the Implicature Base –which is, in turn, well motivated under neo–Gricean principles. Next, we check that (42d) $\rightarrow \mathsf{P}[\geq 4]$:

(44) a. $\mathsf{K}[\geq i] \wedge \neg\mathsf{K}[\geq i+1] \wedge \neg\mathsf{K}[=i] \rightarrow \mathsf{P}[\geq i+1]$ for all $i \in \mathbb{N}$

b. *Proof*. Assume that $i=3$ and suppose that $\neg\mathsf{P}[\geq 4]$. This is equivalent to $\mathsf{K}\neg[\geq 4]$. $\mathsf{K}\neg[\geq 4]$ entails that $\mathsf{K}[=3]$, otherwise the premise $\mathsf{K}[\geq 3]$ would be false. However, $\mathsf{K}[=3]$ contradicts the premise that $\neg\mathsf{K}[=3]$ and so it cannot be that $\mathsf{K}\neg[\geq 4]$. It follows that $\neg\mathsf{K}\neg[\geq 4]$, which is equivalent to $\mathsf{P}[\geq 4]$

The last step is to show that it is not the case that (42d) $\rightarrow \mathsf{P}[=j]$, for all j such that $j > i$. If that were the case, the system would derive TOTAL IGNORANCE instead of PARTIAL IGNORANCE about every value j, which, as we argued above, does not capture the right kind of IIs. In fact, it is the case that $\mathsf{P}[=j]$ and $\mathsf{P}\neg[=j]$ are contingent and logically independent from the Implicature Base. For instance, in the current case, $\neg\mathsf{P}[=4]$ does not contradict any premises nor entails the contradiction of any of them: $\mathsf{K}[\geq 3] \wedge \neg\mathsf{K}[\geq 4] \wedge \neg\mathsf{K}[=3] \wedge \neg\mathsf{P}[=4]$ may or may not be true, depending on the model, but it is neither a contradiction nor a logical truth. Similarly for $\mathsf{K}[\geq 3] \wedge \neg\mathsf{K}[\geq 4] \wedge \neg\mathsf{K}[=3] \wedge \mathsf{P}[=4]$. Notice also that no reference to the notion of symmetry is needed here (see Schwarz and Shimoyama 2011 and Mayr 2013 for discussion).

4.2.2. Results

These are the right kind of IIs. Let us see why. Imagine that (42a) is uttered in a situation where the speaker is uncertain about whether three, five or six boys left, but she knows that exactly four did not leave, and she knows that neither fewer than three nor more than six boys left. In this case, the epistemic state of the speaker S and the assertibility conditions of (42a) are as follows:

(45) a. *Epistemic State of S wrt. to (42a)*:
$\mathsf{K}_S[\geq 3] \wedge \mathsf{K}_S\neg[<3] \wedge \mathsf{I}_S[=3] \wedge \mathsf{K}_S\neg[=4] \wedge \mathsf{I}_S[=5] \wedge \mathsf{I}_S[=6] \wedge \mathsf{K}_S\neg[>6]$

b. *Assertibility conditions of (42a)*:
$\mathsf{K}_S[\geq 3] \wedge \mathsf{K}_S\neg[<3] \wedge \mathsf{P}_S[=3] \wedge \mathsf{P}_S[>3]$

For one, (45a) meets the assertibility conditions defined in (32) above because the speaker considers both three and more than three to be epistemic possibilities compatible with all she knows (in addition to knowing that no number below three is a possibility), and it is the case that $\mathsf{P}_S[=3] \wedge \mathsf{P}_S[>3] \wedge \neg\mathsf{P}_S[<3]$. The assertibility conditions are compatible with the speaker's knowledge that four boys did not leave: $\mathsf{K}_S\neg[=4] \wedge \mathsf{I}_S[>4] \wedge \mathsf{I}_S[=3]$. In fact, the epistemic state of S entails the assertibility conditions of (42a).

These are not, of course, the IIs that the addressee can draw from (42a): In no way could the addressee guess from (42a) that the speaker knows that four boys did not leave; all this example shows is that (42a) is a felicitous sentence in this context. But suppose instead, for completeness, that both speaker and addressee know that four boys did not leave. The epistemic state of the speaker remains the same, whereas we can describe the epistemic state of the addressee as $\mathsf{K}_A\neg[=4] \wedge \mathsf{I}_A[i]$, for some $i \in \{3,5,6\}$. After hearing (42a) the IIs that the addressee can draw are exactly the same as the ones in (42d) above; in other words, the knowledge that four boys did not leave does not interfere with the exact form of the IIs that the addressee is allowed to draw.

4.2.3. No closure under disjunction!

Throughout the paper we have proceeded in parallel comparing the properties of IIs that come with SMNs and disjunction. Here I discuss where the two constructions diverge. The difference is revealed when we consider disjunctions with multiple disjuncts. Recall that, as Alonso-Ovalle (2006) pointed out, in order to derive the right inferences with bigger disjunctions, we need to consider all the smaller disjuncts formed by using two or more of the disjuncts that participate in the original disjunction (see discussion in §2). The calculation I have introduced above shows that in the case of SMNs, this is not the case: We derive the right kind of IIs by looking exclusively at the Implicature Base formed by (i) the Quality Inference and (ii) the Primary Implicatures. In fact, the only alternatives that turn out to be of logical interest –i.e., those that have the power to contradict information in the Implicature Base– are the alternatives that correspond to the material that has been explicitly mentioned: for *at least n*, we have that $[=n]$ and $[\geq n+1]$ are sufficient to derive the right IIs.

But not only is it unnecessary that we consider sub–domain alternatives like the ones required by disjunction, it is obligatory that we do not consider them. The reason is that, even though the meaning of a sentence containing an SMN can be expressed with a multiple disjunction, the IIs that come with SMNs and multiple disjunctions are crucially different. For the sake of the argument, suppose that in order to derive the right IIs for SMNs, we have to find a set of *SA*s whose union carves out the space delimited by the assertion (that is, a set of *symmetric* alternatives; cf. Schwarz 2013; Mayr 2013). Thus, for instance, for an assertion like (42a), we have that the union of the *SA*s $[=3] \vee [\geq 4]$ exhausts the space of the assertion: Because $[=3]$ and $[\geq 4]$ are mutually exclusive, one of them must be true. Then, the right IIs are derived by reasoning specifically about the disjunct $[=3] \vee [\geq 4]$, in a way parallel to the one introduced above.

The appeal of this procedure is that it relies on more or less well-understood facts about disjunction. The problem is that there are more sets of *SA*s that also carve out the meaning of $[\geq 3]$. For instance, the disjunction $([=3] \vee [=4] \vee [\geq 5])$ also covers the meaning of $[\geq 3]$: One of the three disjuncts must be true. The same is true of the multiple disjunct $([=3] \vee [=4] \vee [=5] \vee [\geq 6])$, where one of the four disjunct must be true. And so on.

Schwarz (2013) and Mayr (2013) derive IIs of SMNs by appealing to these sets of symmetric alternatives. By positing *SA*s that cover the meaning of the assertions they manage to provide a suitable disjunctive statement that, after a Gricean reasoning process, delivers the right IIs. Pushing this rationale further, there is no reason why we could not say the same about cases with multiple disjunction, since (i) they also cover the meaning of the assertion, and (ii) they are also amenable to Gricean reasoning.

As a case study, consider the aforementioned disjunct $([=3] \vee [=4] \vee [\geq 5])$ as an *SA* of the assertion in (42a). If we apply exactly the same reasoning as with disjunction, we get the following set of *SA*s:

(46) a. STRONGER ALTERNATIVES:

$$\left\{\begin{array}{l} \big([=3] \vee [=4] \vee [\geq 5]\big), \\ \big([=3] \vee [=4]\big), \big([=3] \vee [\geq 5]\big), \big([=4] \vee [\geq 5]\big), \\ [=3], [=4], [\geq 5] \end{array}\right\}$$

b. PRIMARY IMPLICATURES:

$$\left\{\begin{array}{ll} \neg\mathsf{K}\big([=3] \vee [=4] \vee [\geq 5]\big) & \wedge \\ \neg\mathsf{K}\big([=3] \vee [=4]\big) \wedge \neg\mathsf{K}\big([=3] \vee [\geq 5]\big) \wedge \neg\mathsf{K}\big([=4] \vee [\geq 5]\big) & \wedge \\ \neg\mathsf{K}[=3] \wedge \neg\mathsf{K}[=4] \wedge \neg\mathsf{K}[\geq 5] & \end{array}\right\}$$

c. IMPLICATURE BASE:

$$\left\{\begin{array}{ll} \mathsf{K}[\geq 3] & \wedge \\ \neg\mathsf{K}\big([=3] \vee [=4] \vee [\geq 5]\big) & \wedge \\ \neg\mathsf{K}\big([=3] \vee [=4]\big) \wedge \neg\mathsf{K}\big([=3] \vee [\geq 5]\big) \wedge \neg\mathsf{K}\big([=4] \vee [\geq 5]\big) & \wedge \\ \neg\mathsf{K}[=3] \wedge \neg\mathsf{K}[=4] \wedge \neg\mathsf{K}[\geq 5] & \end{array}\right\}$$

The problem is visible now: the new Implicature Base entails that, for all the speaker knows, $[=4]$ is a live possibility for her. To see why, suppose that $\neg\mathsf{P}[=4]$ or, equivalently, $\mathsf{K}\neg[=4]$. This entails the truth of $\mathsf{K}([=3] \vee [\geq 5])$, since, excluded $[=4]$ as an epistemic possibility, either $[=3]$ or $[\geq 5]$ must be true. But of course, that $\mathsf{K}([=3] \vee [\geq 5])$ contradicts the premise that $\neg\mathsf{K}([=3] \vee [\geq 5])$ in the Implicature Base, and so we must conclude that $\mathsf{P}[=4]$ is a logical truth that follows from (46c). Applying this reasoning recursively to all the possible disjuncts that (i) constitute *SA*s and (ii) cover the meaning of the assertion, we derive an inference of TOTAL IGNORANCE, precisely the kind of II that we want to avoid.

The calculus defended in this paper does not run into this problem, because symmetry is created only for the two relevant *SA*s for which an II can be derived. Thus, the only disjunction that covers the meaning of the assertion is the one corresponding to the material that has been linguistically mentioned. However, by relying solely on the notion of symmetric alternatives that cover the meaning of the assertion we get the wrong IIs. So, in order to make sure that we do not generate unattested IIs, we must neglect *SA*s formed by multiple disjuncts.

The conclusion is that the criteria to choose the right alternatives for SMNs cannot be simply to find a ‘symmetric’ disjunctive statement –i.e., a disjunction of mutually exclusive alternatives that, together, exhaust the meaning of the assertion– because there may be a variety ways to do so with multiple disjuncts. Despite the possible equivalence between SMNs and disjunctions with multiple disjuncts, SMNs cannot be taken to be disjunctions at any level. The difference between disjunction and SMNs is simply that SMNs behave like simple disjunctions with two disjuncts, and so no sub–domain alternatives can be invoked. The IIs that we derive here are simply a function of basic epistemic logic and a neo–Gricean reasoning process. By providing the right source in terms of Horn–Sets for the substitution algorithm to apply, and a basic notion of *SA* in terms of asymmetric entailment, these ingredients suffice to derive the right IIs.

4.3. *The case of* at most

Although Schwarz (2013) does not talk about *at most n*, an extension of this account, limited to the derivation of IIs, goes as follows. The Horn–Sets that we have to consider now are the following two:

(47) a. Horn–Set$_1$: {*at most, exactly*}

b. Horn–Set$_2$: $\{1,2,3,4,...\}$

The substitution mechanism remains the same as the one considered before in (40). An important difference is that with *at most* the logical entailment pattern of the alternatives is reversed with respect to the ones in (41). With *at most*, the *SA*s that we need to consider correspond to those values that are less or equal than the value mentioned in the assertion.[16]

(48) a. ASSERTION: $[\leq 3]$ (e.g., *at most three boys left*)

b. $SA([\leq 3]) = \{[\leq 2],[\leq 1],[=3],[=2],[=1]\}$

c. PRIMARY IMPLICATURES: $\neg\mathsf{K}_S[\leq 2] \wedge \neg\mathsf{K}_S[=3] \wedge \neg\mathsf{K}_S[=2] \wedge \neg\mathsf{K}_S[=1]$

d. IMPLICATURE BASE: $\mathsf{K}_S[\leq 3] \wedge \neg\mathsf{K}_S[\leq 2] \wedge \neg\mathsf{K}_S[=3] \wedge \neg\mathsf{K}_S[=2] \wedge \neg\mathsf{K}_S[=1]$

As before, the Implicature Base in (48d) entails that both *SA*s $[=3]$ and $[\leq 2]$ constitute live epistemic possibilities compatible with all the speaker knows. We have seen the proof many times: Suppose that $\neg\mathsf{P}[=3]$. This is equivalent to $\mathsf{K}\neg[=3]$. Together with $\mathsf{K}[\leq 3]$, $\mathsf{K}\neg[=3]$ entails that $\mathsf{K}[\leq 2]$ is true, but this contradicts the premise that $\neg\mathsf{K}[\leq 2]$, and so we conclude that $\mathsf{P}[=3]$. Assume now that $\neg\mathsf{P}[\leq 2]$. This is equivalent to $\mathsf{K}\neg[\geq 2]$, which, together with $\mathsf{K}[\leq 3]$ entails that $\mathsf{K}[=3]$ must be true. This conclusion contradicts the Primary Implicature that $\neg\mathsf{K}[=3]$, and we must conclude that $\mathsf{P}[\leq 2]$.

There are no surprises in the derivation, and the IIs that we derive are exactly as desired. For instance, suppose that (48a) is uttered in a situation where the speaker is uncertain about whether one or three boys left, but she knows that it is not the case that exactly two boys left and she knows that neither less than one nor more than three boys left. In this case, the epistemic state of the speaker *S* and the assertibility conditions of (48a) are the following:

(49) a. *Epistemic State of S wrt. to (48a)*:
$\mathsf{K}_S[\leq 3] \wedge \mathsf{K}_S\neg[>3] \wedge \mathsf{I}_S[=3] \wedge \mathsf{K}_S\neg[=2] \wedge \mathsf{I}_S[=1] \wedge \mathsf{K}_S\neg[>1]$

b. *Assertibility conditions of (48a)*:
$\mathsf{K}_S[\leq 3] \wedge \mathsf{K}_S\neg[>3] \wedge \mathsf{P}_S[=3] \wedge \mathsf{P}_S[<3]$

The assertibility conditions for *at most* defined earlier in (33) are met: The speaker considers both three and less than three to be epistemic possibilities compatible with all she knows, in addition to knowing that no value above three is a possibility. These assertibility conditions are compatible with the speaker's epistemic state, in particular with the knowledge that exactly two boys did not leave.

[16]For simplicity, I have excluded the proposition corresponding to the number 0.

5. Comparison to other approaches

In this section, I will compare how the approach advocated for in this paper fares when compared to other accounts of SMNs. The focus will be solely on the derivation of IIs, so I will set aside other considerations, such as the behavior of superlatives in embedded contexts. For reasons of space, I will only discuss two of the most influential proposals in the literature. The goal is to show that paying attention to IIs provides a further tool to assess the adequacy of more general theories of SMs. And, as we will see, the kind of IIs that we expect with SMNs cannot be easily captured following those proposals.

5.1. Geurts and Nouwen (2007)

A well–known proposal is Geurts and Nouwen's (2007) modal analysis, where *at least n* is suggested to mean *certainly n and possibly more*. The focus of their paper is on the interaction between superlatives and modal verbs, and so they do not focus on the IIs predicted by their proposal. Only in Geurts et al. (2010:134) they point out that IIs may be derived pragmatically, since *possible* may imply *not certain*, very much in line with the account defended here. In fact, given their lexical entries for SMNs, the IIs they predict are exactly the ones we have defended here:

(50) a. $[\![\text{At least n A's are B}]\!] = \Box\exists x[A(x) \wedge |x| = n \wedge B(x)] \bigwedge \Diamond\exists x[A(x) \wedge |x| > n \wedge B(x)]$

b. $[\![\text{At most n A's are B}]\!] = \Diamond\exists x[A(x) \wedge |x| = n \wedge B(x)] \bigwedge \neg\Diamond\exists x[A(x) \wedge |x| > n \wedge B(x)]$

The interesting consequence of their analysis is that it is not tenable if one considers non–entailing scales, as the authors themselves acknowledge.

(51) a. Sue won at least a bronze medal.

b. $[\![(51a)]\!] = \Box\exists x[win(S,x) \wedge BM(x)] \bigwedge \Diamond\exists x[win(S,x) \wedge x \rhd BM]$

If we apply the lexical entry (50a) to (51b) the result is a contradiction: It is necessary that Sue won a bronze medal, and it is possible that Sue won some other more valuable metal. For this reason, they have to provide a second lexical entry for non–entailing scales.

(52) a. Given a proposition α and a set of alternative propositions of α, $[\alpha]_A$ ordered according to some salient order $\unlhd$ of alternatives,
$[\![\text{AT LEAST } \alpha]\!] = \exists\beta[\alpha \unlhd \beta \wedge \Box\beta] \wedge \exists\gamma[\alpha \unlhd \gamma \wedge \Diamond\gamma]$.

b. $\Box[win(S,bronze) \vee win(S,silver) \vee win(S,gold)] \bigwedge \Diamond[win(S,silver) \vee win(S,gold)]$

The interpretation of (51a) is now as in (52b). The problem with (52b) is that it predicts the wrong kind of IIs. As it is familiar from the literature on Free Choice disjunction, statements of the form $\Box(p \vee q)$ trigger the inference that every disjunct must be a possibility:

(53) a. You must write a paper or a review $\Box(p \vee r)$

b. $[\![53a]\!] \rightsquigarrow$ *you can write a paper or you can write a review.* $\Diamond p \wedge \Diamond r$

This inference is very difficult to cancel, if not impossible. Thus, including a disjunctive statement as part of the semantic import of superlatives makes it equivalent in the relevant respect to a disjunction like (53b). As a consequence, we expect that after hearing a sentence like (51a), the addressee will infer that Betty could have won any of the three medals.

(54) a. $\Box[win(S, bronze) \vee win(S, silver) \vee win(S, gold)]$

b. $[\![(54a)]\!] \rightsquigarrow \Diamond[win(S, bronze) \wedge win(S, silver) \wedge win(S, gold)]$

Given the semantics in (52b) and the truth conditions in (54a), (54b) is an inference that goes through independently of the epistemic state of the agents involved in the conversation. Notice, however, that (51a) is felicitous in situations where the speaker knows that Sue could not have won the gold medal: For instance, the speaker may know that Betty won the race and that Sue and Mary were the next ones crossing the finish line, but she ignores in what order. This is a situation of PARTIAL IGNORANCE. For these cases, Geurts and Nouwen (2007) wrongly predict that the addressee will always draw the inference that Sue may have won the gold medal. This strongly contrasts with the case of disjunction: (55b) contradicts the statement that $\neg\Diamond[win(S, gold)]$ and so the sentence is bad. However, this is not a problem for (55a).

(55) Sue didn't win the gold medal,

a. but she won at least the bronze medal.

b. #but she won the bronze, silver or gold medal.

If Geurts and Nouwen (2007) are right, one would still have to explain why (55b) is not an acceptable follow up to the previous sentence, since both (55a) and (55b) denote a statement of the form $\Box(bronze \vee silver \vee gold)$. Although this argument alone may not be fatal for Geurts and Nouwen's (2007) approach, they certainly predict the wrong inferences about what it can be inferred about what the speaker knows after she uttered a sentence containing an SM.[17]

5.2. Coppock and Brochhagen (2013b)

Couched in the framework of inquisitive semantics, Coppock and Brochhagen (2013b) develop an account on which superlative modifiers, rather that being disjunctive themselves, share with disjunction the property of being *interactive*, that is, the property that they denote more than one possibility. In inquisitive semantics, denotations are represented as sets of possible worlds (possibilities) corresponding to the set of possible answers to the QuD. For instance, the sentence *at least three boys came* can denote the possibilities that three boys came, that four boys came, etc. In this case, $[\![$*at least three boys left*$]\!]$ denotes the set $\{p_n | n \geq 3\}$. This set is then further constrained by the *information state* of the speaker, the set of epistemically accessible worlds for a speaker. This feature makes Coppock and Brochhagen's (2013b) account compatible with PARTIAL IGNORANCE: If the addressee knows that five boys did not come, then he will factor out worlds where the possibility of five boys coming is alive.

The first thing to notice about this account is that the authors do not provide any independent evidence for the claim that superlative modifiers are interactive –i.e., for the claim that they

[17] Rather than as a criticism, these remarks on Geurts and Nouwen (2007) should be taken as a reminder that we may learn a great deal about SMs just by looking at the IIs they convey. I refer the reader to Mendia (2015) for further discussion of IIs with superlatives that go beyond the numeral case.

denote at least two epistemic possibilities. In turn, Schwarz (2013, 2014), and therefore this paper too, have a clear advantage over both Büring's (2007) and Coppock and Brochhagen's (2013b) proposals in that the pragmatic properties that disjunction and SMs share can be derived from well–motivated independent principles that are already present in the theory.

Second, what seems to be a good feature of this approach, the ability to derive PARTIAL IGNORANCE, is too permissive, once we consider other properties of IIs. Assume, for example, that I know that no more than three boys came, but I do not know how many exactly. In this context, the sentence *at least fours boys came* is pragmatically infelicitous, since it conveys that it is possible that exactly fours boys came. Coppock and Brochhagen (2013b) do not predict this oddness: in this context, *at most three boys left* and *at most four boys left* have the same denotation. All *at most* needs in order to be felicitous is that there be at least two different possibilities in its denotation that are candidates to be the actual world. In this case, both sentences meet this criterion since for all the speaker knows, it could be that three, two or one boys left.

The main problem is that it is not specified which possibilities should *at most* denote: As long as there are two, the sentence is predicted to be felicitous, and it will denote those possibilities that are compatible with the speaker's information state. In the example at hand, both sentences *at most* {*three/four*} *boys left* will denote the same set of possibilities, and so they are predicted to be equivalent. In short, *SM n A's B* does not entail that *exactly n A's B* is an epistemic possibility for the speaker, which is one of the three conditions that makes an SM assertible (see §3.2). In Coppock and Brochhagen (2013a), the authors propose an additional pragmatic principle, the Maxim of Depictive Sincerity, which amounts to say, roughly, that if a speaker "highlights" a possibility, then she considers that possibility compatible with her own knowledge. In the case of SMs, they further assume that *SM n A's B* highlights *exactly n A's B*. In the account defended in this paper, we can dispense with this extra pragmatic principle and rely on a better understood neo–Gricean reasoning process.

6. *Conclusions*

The main conclusion of the paper is that IIs that come with disjuncts and SMNs must be derived by considering different sets of alternative propositions. In the case of multiple disjuncts, every possible disjunction of at least two disjuncts from the original disjunction have to be considered (Alonso-Ovalle 2006). In the case of SMNs, on the other hand, the only alternatives that matter are the alternatives that have been explicitly mentioned, as originally argued by Büring (2007).

The upshot is that we should care about the kind of IIs that we want SMNs to come with. We should not be misguided by different but seemingly equivalent formulations of what the inferences exactly are: For instance, the fact that SMNs can be equivalent to a disjunction with multiple disjuncts should not be taken as evidence that we can draw inferences from the latter. Different formulations of exactly what IIs SMs come with may lead to different predictions. In this paper, I compared two such predictions and concluded that IIs of SMNs are better characterized after Büring's (2007) proposed meaning for SMNs. In addition, I attempted to show that we do not need to assume a conventionalized meaning or a syncategorematic treatment of SMNs, unlike Büring (2007). Instead, I argued that we can account for the appropriate IIs by carefully choosing the right pieces from other relatively well–studied phenomena and putting them together in the right way. Concretely, IIs that come with SMNs can be accounted for in a purely neo–Gricean framework, using the double Horn–Set method first advanced in Schwarz

(2013) and Mayr (2013). The rest follows from the properties of Hintikka's (1962) epistemic operators K and P and a basic epistemic logic.

It is an open question whether this approach can be extended to cover other cases of superlatives, like the propositional case and non–entailing scales. In Mendia (2015) I show that similar IIs arise with superlatives modifiers modifying all shorts of constituents. I also tried to make a case for understanding superlative modifiers as focus sensitivity operators, in Krifka's (1999) vein. If it turns out that a focus–sensitivity approach is tenable, the next step is to explore its consequences for embedding contexts like negation and overt modal operators.

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